IV. *A Letter from Mr.* Colin Mac Laurin, *Professor of Mathematicks at* Edinburgh, *and* F.R.S. *to* Martin Folkes, *Esq; V. Pr. R. S. concerning Æquations with impossible Roots.*

S I R,

I Wrote to you last Winter, that I had thought of a very easie and simple way of demonstrating Sir *Isaac Newton*'s Rule, by which it may be often discover'd when an Æquation has impossible Roots. This Method requiring nothing but the common Algebra, and being founded on some obvious Properties of Quantities demonstrated in the following *Lemmata*, without having recourse to the Consideration of any Curve whatsoever, which does not seem so proper a Method in a Matter purely Algebraical, I hope it will not be unacceptable.

Lemma 1. The Sum of the Squares of two real Quantities is always greater than twice their Product. Thus a^2+b^2 is greater than $2ab$; because the Excess a^2+b^2-2ab is eqaal to $\overline{a-b}^2$, and therefore is Positive; since the Square of any real Quantity, Negative or Positive, is always Positive.

Lemma 2. The Sum of the Squares of three real Quantities is always greater than the Sum of the Products, that can be made by multiplying any two of them. Thus $a^2+b^2+c^2$ is always greater than $ab+ac+bc$; for 'tis plain, that the Excess $a^2+b^2+c^2-ab-ac-bc$

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$= \frac{2a^2+2b^2+2c^2-2ab-2ac-2bc}{2} = \frac{a^2-2ab+b^2+a^2-2ac+c^2+b^2-2bc+c^2}{2} = \frac{\overline{a-b}^2+\overline{a-c}^2+\overline{b-c}^2}{2}$ that is, half the Sum of the Squares of the Differences of the Quantities a, b, c: But ſince theſe Squares are Poſitive, it follows, that the Exceſs of $a^2+b^2+c^2$ above $ab+ac+bc$ is Poſitive, and that the Sum of the Squares of three Quantities muſt be greater than the Sum of the Products made by multiplying any two of them.

Lemma 3. The triple Sum of the Squares of four Quantities is greater than the double Sum of the Products, that can be made by multiplying any two of them; for $3a^2+3b^2+3c^2+3d^2-2ab-2ac-2ad-2bc-2bd-2cd = a^2-2ab+b^2+a^2-2ac+c^2+a^2-2ad+d^2+b^2-2bd+d^2+b^2-2bc+c^2+c^2-2cd+d^2 = \overline{a-d}^2+\overline{a-c}^2+\overline{a-d}^2+\overline{b-c}^2+\overline{b-d}^2+\overline{c-d}^2$, the Sum of the Squares of the Differences of the four Quantities a, b, c, d. Therefore $3a^2+3b^2+3c^2+3d^2$ is greater than $2ab+2ac+2ad+2bc+2bd+2cd$, the Exceſs being always Poſitive.

Lemma 4. Let the Number of the Quantities a b, c, d, e, &c. be m, the Sum of their Squares A, and the Sum of the Products made by multiplying any two of them B. Then ſhall $\frac{\overline{m-1}\times A}{2}$ be always greater than B.

For by adding together the Squares of the Differences $a-b$, $a-c$, $a-d$, $b-c$, $b-d$, $c-d$, &c. you add a^2 as often to it ſelf as there are Quantities more than a; the ſame is true of b^2, c^2, d^2, e^2, &c. But the Rectangles $-2ab$, $-2ac$, $-2ad$, $-2bc$, $-2bd$, &c. ariſe but once each. Therefore the Sum of all the Squares $\overline{a-b}^2$, $\overline{a-c}^2$, $\overline{b-c}^2$, $\overline{b-d}^2$, &c. $= \overline{m-1}\times a^2+\overline{m-1}\times b^2+\overline{m-1}\times c^2$, &c. $-2ab-2ac-2bc$, &c. $= \overline{m-1}\times A - 2B$. But $\overline{a-b}^2$

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$+\overline{a-c}^2+\overline{a-d}^2$, &c. is always a Positive Quantity; therefore $\overline{m-1}\times A-2B$ is Positive, and consequently $\frac{\overline{m-1}\times A}{2}$ greater than B.

Cor. It appears from the Demonstration that the Excess of $\overline{m-1}\times A$ above $2B$ is always equal to the Sum of the Squares of the Differences of the Quantities $a, b, c, d,$ &c. and that when the Quantities a, b, c, d &c. are all equal, then $\overline{m-1}\times A-2B=0$, and with this restriction the preceding *Lemmata* must be understood.

It is to be observed, that tho' we have supposed in these *Lemmata* the Quantities $a, b, c, d,$ &c. Positive, they are *a fortiori* true of Negative Quantities, whose Squares are the same as if they were Positive, while the Sum of their Products is either the same, or less than it would be, were they all Positive.

PROP. I.

In a Quadratic Æquation that has its Roots real, the Square of the second Term must be always greater than the quadruple Product of the third and first Terms.

Let the Roots of the Quadratic Æquation be represented by $+a$ and $+b$; and if x be the unknown Quantity, then shall
$$\begin{array}{l} x^2-ax+ab=0 \\ \quad\;\, -bx \end{array}$$

Now since a^2+b^2 is greater than $2ab$, by *Lemma* 1, therefore a^2+b^2+2ab is greater than $4ab$; therefore $\overline{a+b}^2\times x^2$, the Square of the second Term, will be greater than $4ab\times x^2$ the Quadruple Product of the first and third Terms.

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PROP. II.

In any Cubic Æquation, all whose Roots are real, the Square of the second Term is always greater than the triple Product of the first and third.

If the Cubic Æquation has all its Roots real, they may be represented with their Signs by $a, b, c,$ and the Æquation will be expressed thus:

$$\begin{aligned} y^3 - ay^2 &+ aby - abc = o \\ -by^2 &+ acy \\ -cy^2 &+ bcy \end{aligned}$$

But by *Lemma* 2, $a^2+b^2+c^2$ is always greater than $ab+ac+bc$; and consequently adding $2ab+2ac+2bc$ to both sides, $a^2+b^2+c^2+2ab+2ac+2bc$ $(=\overline{a+b+c}^2)$ will be greater than $3ab+3ac+3bc$; and therefore $\overline{a+b+c}^2 \times y^4$ must be greater than $\overline{3ab+3ac+3bc} \times y^4$, that is, the Square of the second Term must be greater than the triple Product of the first and third Terms.

Cor. 1. In general, it appears from the Demonstration, that the Square of the Sum of three real Quantities, $\overline{a+b+c}^2$ is always greater than the triple Sum of all the Products, that can be made by multiplying any two of them into one another.

Cor. 2. It follows from the Proposition, that when the Square of the second Term is not greater than the triple Product of the first and third Terms, the Roots of the Æquation cannot be all real; but two of them must be impossible: And this plainly coincides with one Part of Sir *Isaac Newton*'s Rule for discovering when the Roots of Cubic Æquations are impossible.

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He desires you may write above the middle Terms of the Æquation the Fractions $\frac{1}{3}$, $\frac{1}{3}$ as in the Margin; and placing the Sign + under the first and last Term, he multiplies the Square of the second Term by the Fraction $\frac{1}{3}$ that is above it; and if the Product is greater than the Product of the adjacent Terms, he places + under the second Term; but if that Product is less, he places — under the second Term, and says, there are as many impossible Roots as changes in the Signs. Now by this Proposition, if $p^2 x^4$ is not greater than $3\, q x^4$, or $\frac{1}{3}\, p^2 x^4$ greater than $q x^4$, the Roots cannot be all real. The same Supposition makes two Changes in the Signs, whatever Sign you place under the third Term, since the Signs under the first and last are both +; and therefore this Proposition demonstrates the first Part of Sir *Isaac Newton*'s Rule, as far as it relates to Cubic Æquations.

$$\begin{array}{cccc} & \frac{1}{3} & \frac{1}{3} & \\ x^3 & +\,px^2 & +\,qx & +\,r=0 \\ + & - & * & + \end{array}$$

Cor. 3. If the second Term is wanting in a Cubic Æquation, and the third is Positive, two of the Roots of the Æquation must be impossible: For the Square of the second Term (equal to nothing in this Case) will be less than the triple Product of the adjacent Terms. But this will better appear from considering that, when the second Term vanishes in an Æquation, the Positive and Negative Roots are equal, and when added together, destroy each other. Suppose the Roots to be $+a$ and $-b, -c$; then in this Case $a = +b+c$, and the Coefficient of the third Term will be $-ab-ac+bc = -b^2-2\,bc-c^2+bc = -b^2-bc-c^2$, and consequently Negative. Or, if you suppose two Roots Positive and one Negative, let them be $-a, +b, +c$, then the Coefficient of the third Term will be still $-b^2-bc-c^2$. Therefore when the Roots are real, the

the Coefficient of the third Term is negative; and if the Coefficient of the third Term is not affected with a negative Sign, it is a Proof that two of the Roots are Impossible.

PROP. III.

In any Cubic Æquation, all whose Roots are real, the Square of the third Term must be greater than the triple Product of the second and fourth Terms.

In the same Cubic Æquation, whose Roots are a, b, c, the Square of the third Term is $\overline{ab+ac+bc}^2$, the Product of the second and fourth Terms is $a^2bc+ab^2c+abc^2$, as is plain from the Inspection of the Æquation; and it is obvious that $a^2bc+ab^2c+abc^2$ is the Sum of the Products of any two of the Terms ab, ac, bc; and therefore by *Corol.* 1 *Prop.* 2. the Square of the Sum of these Terms, that is, $\overline{ab+ac+bc}^2$ must be greater than $3a^2bc+3ab^2c+3ac^2b$. So that $\overline{ab+ac+bc}^2 \times y^2$ must be greater than $\overline{3a^2bc+3ab^2c+3ac^2b} \times y^2$; that is, the Square of the third Term must be greater than the triple Product of the second and fourth Terms.

Cor. 1. It follows from the Demonstration, that $\overline{ab+ac+bc}^2$ is always greater than $3abc \times \overline{a+b+c}$.

Cor. 2. If the Square of the third Term is found to be less than the triple Product of the second and fourth Terms, then the Roots of the Æquation cannot be all real Quantities; and this concludes with the second Part of Sir *Isaac Newton*'s Rule for finding when the Roots of a Cubic Æquation are impossible: For this Case gives — to be placed under the third Term, and

$$x^3 + \overset{\frac{1}{3}}{p}x^2 + \overset{\frac{1}{3}}{q}x + r = 0$$
$$+\;*\quad\;\; -\;\; +$$

consequently two Changes of the Signs, whatever Sign is placed under the second Term.

Schol. After the same manner, it may be demonstrated, that in a Cubic Æquation, whose Roots are all real, if the second Term is wanting, the Cube of the third Part of the third Term taken positively, is always greater than the Square of half the last Term. Suppose that the Roots of the Æquation are $+a, -b, -c$, or $-a, +b, +c$, and that $a = b + c$, then the second Term in the Æquation will be wanting, and the other Terms will be expressed thus:

$$\begin{array}{l} y^3 * -b^2 y \pm bc \times \overline{b+c} \\ \quad\;\; -bcy \\ \quad\;\; -c^2 y \end{array}$$

The Square of $b-c$ is always positive, since b and c are real Quantities. Suppose it, (*viz.* $b^2 - 2bc + c^2$) equal to D, then $b^2 + bc + c^2 = D + 3bc$, and $\overline{b+c}^2 = D + 4bc$. Therefore $\frac{\overline{b^2+bc+c^2}^3}{27} = \frac{D^3}{27} + \frac{D^2 bc}{3} + Db^2 c^2 + b^3 c^3$, and $b^2 c^2 \times \frac{\overline{b+c}^2}{4} = \frac{Db^2 c^2}{4} + b^3 c^3$. Now 'tis obvious that $\frac{D^3}{27} + \frac{D^2 bc}{3} + Db^2 c^2 + b^3 c^3$ is greater than $\frac{Db^2 c^2}{4} + b^3 c^3$, since D is positive, and bc also positive, b and c being Roots having the same Sign. Therefore the Cube of $\frac{1}{3}$ of the third Term having its Sign changed $\left(= \frac{\overline{b^2+bc+c^2}^3}{27}\right)$ is always greater than the Square of half the last Term $\left(= b^2 c^2 \times \frac{\overline{b+c}^2}{4}\right)$. In the Cubic Æquation $x^3 * + qx + r = 0$, if q be positive, or if it

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be negative and $\frac{+q^3}{27}$ be leſs than $\frac{1}{4}r^2$, it appears that two Roots of the Æquation muſt be impoſſible, from this Corollary, and from *Cor.* 3. *Prop.* 2. taken together.

PROP. IV.

In a Biquadratic Æquation, all whoſe Roots are real Quantities, $\frac{3}{8}$ of the Square of the ſecond Term is always greater than the Product of the firſt and third Terms; and $\frac{3}{8}$ of the Square of the fourth Term is always greater than the Product of the third and fifth Terms.

1. Let the Æquation be $x^4 - px^4 + qx^2 - rx + s = o$; and ſince the Roots are ſuppoſed to be all real, let them be repreſented by a, b, c, d, then $p = a+b+c+d$, and $q = ab+ac+ad+bc+bd+cd$ But it is plain from *Lemma* 3, that $3a^2+3b^2+3c^2+3d^2$ is greater than $2ab+2ac+2ad+2bc+2bd+2cd$; and conſequently by adding $6ab+6ac+6ad+6bc+6bd+6cd$ to both, we ſhall find that $3\times\overline{a+b+c+d}^2$ muſt be greater than $8ab+8ac+8ad+8bc+8bd+8cd$; that is, $3p^2$ greater than $8q$; and therefore $\frac{3}{8}p^2x^6$ greater than qx^6.

2. Since $r = abc+abd+acd+bcd$, and $s = abcd$; and ſince qs is equal to $a^2b^2cd+a^2c^2bd+a^2d^2bc+b^2c^2ad+b^2d^2ac+c^2d^2ab$, which are the Products can be made of any two of the Quantities abc, abd, acd, bcd, whoſe Sum is r multiplied by one another; it follows, that $3r^2$ is always greater than $8qs$: So that $\frac{3}{8}$ of either the Square of the ſecond Term, or of the Square of the fourth Term, muſt al-

ways be greater than the Product of the Terms adjacent to them.

Cor. Multiply either the Square of the second Term, or the Square of the fourth Term of a Biquadratic Æquation by $\frac{3}{8}$, and if the Product does not exceed the Product of the adjacent Terms, some of the Roots of that Æquation must be impossible.

PROP. V.

In an Æquation of any Dimension expressed by m, the Coefficients of the second, third, last, last but one, and last but two Terms, being respectively A, B, E, D, C if the Roots of the Æquation are all real, then shall $\overline{m-1}\times A^2$ always be greater than $2mB$, and $\overline{m-1}\times D^2$ greater than $2mCE$.

1. For supposing the Roots to be a, b, c, d, e, &c. then by *Lemma* 4, shall $\overline{m-1}\times a^2+\overline{m-1}\times b^2+\overline{m-1}\times c^2$ &c. be greater than $\overline{2ab}+\overline{2ac}+\overline{2ad}$, &c. and adding $\overline{2m-2}\times ab+\overline{2m-2}\times ac+\overline{2m-2}\times ad$, &c. to both, the Sum $\overline{m-1}\times a^2+\overline{2m-2}\times ab+\overline{m-1}\times b^2+$ &c. $(=\overline{m-1}\times\overline{a+b+c,\&c.}^2)$ must be greater than $2mab+2mac+2mad$, &c. that is, $\overline{m-1}\times A^2$ must be greater than $2mB$.

2. In general, it follows from this Demonstration, that the Square of the Sum of any Quantities whose Number is (m) multiplied by $m-1$, must be greater than the Sum of all the Products can be made by multiplying any two of them, multiplied by $2m$. But it is easie to see from the Genesis of Æquations, that CE is the Sum of the Products can be made by multiplying any two of the Terms whose Sum is D: From which it follows, that $\overline{m-1}\times D^2$ must be always greater than $2mCE$.

To be continued. V. *A*